Math 261B Thurs. 9/17

$G \quad A = \mathcal{O}(G) \qquad \mathfrak{g} = \mathrm{Lie}(G) = T_e G \subset A^*$

$(\mathfrak{m}_e/\mathfrak{m}_e^2)^* = T_e G \hookrightarrow A^*$

$\xi \in A^* \qquad \xi \mid (\mathfrak{m}_e^2 + K)$

$U = \varinjlim_n (A/\mathfrak{m}_e^n)^* \subset A^* \quad =$ left-invariant differential operators

$\cup\!\mathsf{I}$

$\mathfrak{g} =$ left-invariant vector fields

$[\,,\,]$ is commutator in A^*

$G \hookrightarrow A^* \qquad g \mapsto ev_g$

$\mathrm{Ad}: G \curvearrowright \mathfrak{g}$ is conjugation in A^*

For $G = GL_n \quad (X-I)_{ij}$ generate $\mathfrak{m}_e$ basis of $\mathfrak{m}_e/\mathfrak{m}_e^2$

Dual basis $\partial x_{ij}|_I$ of $T_e G = \mathfrak{g}$

Last time: $\partial x_{ij} \longleftrightarrow E_{ij} \quad \begin{pmatrix} & 1 \\ & 1_j \end{pmatrix} \leftarrow i \qquad \mathfrak{g} \cong M_n = \mathfrak{gl}_n$

$[\,,\,]$ is commutator in M_n

$\mathrm{Ad}: G \curvearrowright \mathfrak{g}$ is conjugation by GL_n on M_n.

$$\begin{pmatrix} * & & \times \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad \neq 0$$

Pick $GL_n \supset B \supset T$ maximal alg. torus

Borel = maximal solvable closed subgroup

$B = (\text{upper } \Delta\text{'ar}) \leq GL_n$

$T = (\text{diagonal}) \cong \mathbb{G}_m^n$

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \quad t_i \neq 0$$

B is defined by equations $X_{ij} = 0$ for $i > j$

$\mathcal{O}(B) = \mathcal{O}(G) / (X_{ij} \quad i > j)$

$T_e B \hookrightarrow T_e G$

$\vdots$ dual

$m_e/m_e^2(B) \leftarrow m_e/m_e^2(G)$

$\mathcal{O}(B) \twoheadleftarrow \mathcal{O}(G)$ basis $(X - I)_{ij}$

$0 \longmapsfrom$ for $i > j$

$R_u(B) = U = (\text{upper uni } \Delta\text{'ar})$

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

∂X_{ij} dual basis of $(m_e/m_e^2)^* = T_e G$

Basis $\{\partial X_{ij} \mid i \leq j\}$ of $\mathfrak{b} = T_e B = \mathrm{Lie}(B)$

$\mathfrak{gl}_n = M_n \qquad \mathfrak{b} = (\text{upper } \Delta\text{'ar})$

U has eq'n $(X-I)_{ij}=0$ for $i \leq j$ $\Rightarrow$ $\mathfrak{u} = \mathrm{Lie}(U) =$ (strictly upper Δ'ar)

T $X_{ij}=0$ for $i \neq j$ $\Rightarrow$ $\mathfrak{t} = \mathrm{Lie}(T)$
$=$ (diagonal matrices)

(For Lie/$\mathbb{C}$ have $\exp: \mathfrak{g} \to G$ is matrix exponential)

$U = R_u(B)$ $\quad T \hookrightarrow B$ $\quad T \xrightarrow{\sim} B/U$ $\quad B = T \ltimes U$

($G \supset B \supset T$ $\quad B = T \ltimes U$ always works)

How $\mathrm{Ad}: T \curvearrowright \mathfrak{gl}_n \cong M_n$ $\quad$ ($T \curvearrowright \mathfrak{g}$)

$$\underline{t} \cdot \underline{x} \quad \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \cdot X \cdot \begin{pmatrix} t_1^{-1} & & \\ & \ddots & \\ & & t_n^{-1} \end{pmatrix}$$

$X = X(T) \supset$ distinguished elements.

$X = E_{ij}$ $\quad \underline{t} \cdot E_{ij} = (t_i/t_j)\, E_{ij}$ $\quad E_{ii}$'s span $\mathfrak{t} = \mathrm{Lie}(T)$

$\mathrm{Ad}: T \curvearrowright \mathfrak{t}$ is trivial since T is abelian $\quad t_i/t_i = 1$

$\underline{t} \mapsto \underline{t}^{0}$ is $0 \in X = X(T)$ $\qquad \mathfrak{t} \subseteq \mathfrak{g}_0$ (maximality of T $\Rightarrow \mathfrak{t} = \mathfrak{g}_0$?)

$$X = X(T) = \mathbb{Z}^n \ni (a_1, \dots, a_n) \mapsto \left(\underline{t} \mapsto t_1^{a_1} \cdots t_n^{a_n}\right)$$

$k \cdot E_{ij}$ has T character $e_i - e_j \in \mathbb{Z}^n$ $\qquad e_i = i^{th}$ unit vector

$$\underline{t}^{e_i - e_j} = t_i / t_j$$

$$R = \{\underline{\text{roots}}\} = \{e_i - e_j \in \mathbb{Z}^n \mid i \neq j\}$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad \mathfrak{g}_\alpha = k E_{ij} \quad \alpha = e_i - e_j$$

$$\mathfrak{g}_0 = \mathfrak{t}$$

$$X \supseteq R$$

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \qquad R_+ = \{\text{positive roots}\} = \{e_i - e_j \mid i < j\}.$$

$$U_- \cdot T \cdot U_+$$

$$\mathfrak{u} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \qquad R = R_+ \sqcup -R_+$$

$N(T)/T = W$ Weyl group $\cong S_n$ $(\subset GL_n)$ ☆

$$\underbrace{\begin{pmatrix} & * & & * & * \\ * & & & & \\ & & * & & \\ & & & & * \end{pmatrix}}_{N(T)}$$

← like a permutation matrix but non-zero entries need not be 1.

W acts on the set of roots R, acts on the various Borels $B \supset T$ ← fixed

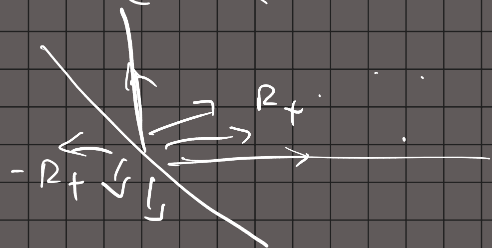
R_+ depends on choice of B,

but all possibilities are symmetric (simply transitively)

under W.

$Q = \mathbb{Z}R \subset X$ $\quad X = \mathbb{Z}^n \quad Q = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum a_i = 0\}$

$Q_+ = \mathbb{N}R_+$



$e_i - e_j \quad i < j$

$(\cdots 1, 0 \cdot\cdot -1\ 0 \cdot\ 0)$

$e_i - e_{i+1} = \alpha_i \quad (0 \cdots 1 -1 \cdots 0) \quad e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots \alpha_{j-1}$

$i = 1, \dots, n-1$ $\quad\quad \mathbb{N}$

'minimal' pos. roots : simple roots $\Delta \subset R_+$

α_i are lin. independent and $Q_+ = \mathbb{N}\Delta$

$\mathbb{R}Q_+ = \mathbb{R}\Delta$

they lie on the extreme rays.

X, R, R_+, Δ

SL_n vs. GL_n $\quad SL_n \subset GL_n$

$\cup$

$B \supset T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$

$\det(x) = 1 \qquad z = 1$

$t_1 \cdots t_n = 1$

$GL_n \xrightarrow[\det]{} \mathbb{G}_m$

$\mathfrak{sl}_n \subset \mathfrak{gl}_n$

$\parallel$

trace 0 matrices

$0 \to T(SL_n) \longrightarrow T(GL_n) \longrightarrow \mathbb{G}_m \to 0$

$t \mapsto (t_1 \cdots t_n)$

$\chi_{(1,\dots,1)}$

$\partial x_{ij} (t_1 \cdots t_n)\big|_I = 1$ for each x_{ii}

$\mathfrak{gl}_n \xrightarrow[d]{} K$

trace

$E_{ii} \xrightarrow[d]{} 1$

$G \to H$

$e \quad e$

$T_e \;\; d_e f \;\; T_e$

$X(SL_n) \qquad X(GL_n) \quad (1,\dots,1) \in \mathbb{Z}^n = X$

$0 \leftarrow \mathbb{Z}^n / \mathbb{Z}\cdot(1,\dots,1) \leftarrow \mathbb{Z}^n \longleftarrow \mathbb{Z}$

$(1,\dots,1)$

R, R_+, Δ same as before, but in $\mathbb{Z}^n / \mathbb{Z}\cdot(1,\dots,1)$

$e_i - e_j$

$e_1 - e_2 \quad e_2 - e_1$

(PGL_n coming up)

SL_2:

$GL_2 \quad \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

$\cup$

$\mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

$E_{11} - E_{22} \quad E_{12} \quad E_{21}$

Coroots $\alpha \in R$ $\qquad [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \to \mathfrak{g}_0$

$$k \cdot h_\alpha$$

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus k \cdot h_\alpha \text{ is a Lie subalgebra of } \mathfrak{g}$$

$$\cong \mathfrak{sl}_2$$

These come from homomorphisms $\qquad$ SL_2 simply connected

$$SL_2 \to G.$$

(Easy for Lie groups / $\mathbb{C}$, have work for alg. gps).

$$j \downarrow$$

$$GL_n : \quad \begin{array}{c} \\ j \to \end{array} \begin{pmatrix} 1 & & & & & \\ & a & & b & & \\ & & 1 & & & \\ & & & \ddots & & 0 \\ & c & & & d & \\ 0 & & & & & 1 \end{pmatrix} \leftarrow i \quad \longleftarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\uparrow i \qquad\qquad SL_2$$

$$\begin{pmatrix} t & * \\ & t^{-1} \end{pmatrix}$$

$$t \in \mathbb{G}_m$$

$$T(SL_2) = \mathbb{G}_m$$

$$SL_2 \to G$$

$$\boxed{\text{coroots}}$$

$$\mathbb{G}_m \to T(G) \text{ is a } \underline{\text{cocharacter}} \text{ of } T \quad \in X(T)^*$$

$$\cong (\mathbb{Z}^n)^*$$

For root $\underset{\alpha}{\overset{\parallel}{e_i - e_j}}$ $\qquad t \longmapsto \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & t & & & & \\ & & & \ddots & & & \\ & & & & t^{-1} & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix} \quad \dashrightarrow \quad t^{a_i - a_j}$

$$\frac{t^\alpha}{t_1^{a_1} \cdots t_n^{a_n}}$$

$\uparrow$

$t_i = t \quad t_j = t^{-1}$

$t_k = 1$

Identify $(\mathbb{Z}^n)^* = \mathbb{Z}^n$ by standard pairing $\quad (\mathbb{Z}^n)^*$ has basis ε_i dual to e_i

$\alpha = e_i - e_j \quad \longleftrightarrow \quad$ coroot $\quad \alpha^\vee = \varepsilon_i - \varepsilon_j \qquad \langle \varepsilon_i, e_j \rangle = \delta_{ij}$

$$\begin{array}{cccccc} & & R^+ & & & \alpha^\vee \\ & & \cap & & & \cap \\ & & & & & X^* \\ (X, & X^*, & R, & R^\vee &) & \\ & & \cap & \cap & & \\ & & X & X^* & & \end{array}$$

Cartan matrix : $\langle \alpha_i^\vee, \alpha_j \rangle$

$$\langle \varepsilon_i - \varepsilon_{i+1}, e_j - e_{j+1} \rangle = \begin{cases} 0 & \text{if } |i-j| > 1 \\ -1 & \text{if } |i-j| = 1 \\ 2 & \text{if } i = j \end{cases}$$

$$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \quad (n-1) \times (n-1)$$

$\longleftrightarrow \quad \circ\!-\!\!\!-\!\circ\!-\!\!\!-\!\circ \cdots -\!\!\!-\!\circ$

A_{n-1}

$PGl_n = Gl_n / K \cdot 1 \qquad 1 \to \mathbb{G}_m \to Gl_n \twoheadrightarrow PGl_n \to 1 \qquad (1, \dots, 1)$

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \Big/ \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix}$$

$X(PGl_n) \qquad t \mapsto \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix}$

$1 \to \mathbb{G}_m \to T \to T(PGl_n) \to 1$

$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}^n \leftarrow \{(a_1, \dots, a_n) \mid \sum a_i = 0\} \ni 0$

$a_1 + \dots + a_n \leftarrow\!\shortmid (a_1, \dots, a_n)$

$R, R^\vee$ still $\uparrow$

$e_i - e_j$ "

$$\begin{matrix} R \subset & X(PGl_n) & & X(PGl_n)^* & R^\vee \\ \| & \| & R \quad R^\vee & \| & \| \\ R^\vee & X(SL_n)^* & & X(SL_n) & R \end{matrix}$$

$\mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$

$(X, X^*, R, R^\vee)$ for Gl_n is self-dual

for SL_n dual to PGl_n

$PGl_n = (SL_n)^L$ is Langlands Dual